
Answers and Hints

14.3 Labor as Gift Exchange

- (a) Choosing w and N to maximize profits gives the first order conditions

$$\pi_w(w, N) = [f'(eN)e' - 1]N = 0 \quad (14.1)$$

$$\pi_N(w, N) = f'(eN)e - w = 0. \quad (14.2)$$

Solving these equations gives the Solow condition;

- (b) The second partials are:

$$\begin{aligned} \pi_{ww} &= [f''Ne'^2 + f'e'']N < 0, & \pi_{NN} &= f''e^2 < 0, \\ \pi_{wN} &= f''Nee' + f'e' - 1 = f''Nee' < 0. \end{aligned}$$

It is easy to check that the second order conditions are satisfied: $\pi_{ww} < 0$, $\pi_{NN} < 0$, and $\pi_{ww}\pi_{NN} - \pi_{wN}^2 > 0$.

- (c) To show that $dw/dz > 1$, differentiate the first order conditions (14.2) totally with respect to w and N :

$$\begin{aligned} \pi_{ww} \frac{dw}{dz} + \pi_{wN} \frac{dN}{dz} + \pi_{wz} &= 0 \\ \pi_{Nw} \frac{dw}{dz} + \pi_{NN} \frac{dN}{dz} + \pi_{Nz} &= 0 \end{aligned} \quad (14.3)$$

Solving these two equations in the two unknowns dw/dz and dN/dz , we find

$$\frac{dw}{dz} = -\frac{\pi_{NN}\pi_{wz} - \pi_{Nw}\pi_{Nz}}{\pi_{NN}\pi_{wz} - \pi_{Nw}^2}. \quad (14.4)$$

But we also calculated directly that

$$\begin{aligned} \pi_{wz} &= -[f''Ne'^2 + f'e'']N = -\pi_{ww}, \\ \pi_{Nz} &= -f'e' - f''Nee' = -f'e' - \pi_{Nw}. \end{aligned}$$

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Substituting these values in (14.4), we get

$$\frac{dw}{dz} = 1 - \frac{\pi_{Nw} f' e'}{\pi_{NN} \pi_{wz} - \pi_{Nw}^2},$$

and the fraction in this expression is negative (the denominator is positive by the second order conditions, while $\pi_{Nw} < 0$ and $f', e' > 0$).

Since $dw/dz > 1$, it follows from the chain rule that

$$\begin{aligned} \frac{de}{dz} &= e' \left[\frac{dw}{dz} - 1 \right] > 0, \\ \frac{dN}{dz} &= \frac{-\pi_{wz} - \pi_{ww} \frac{dw}{dz}}{\pi_{wN}} = \frac{\pi_{ww}}{\pi_{wN}} \left(1 - \frac{dw}{dz} \right) < 0 \quad [\text{from (14.3) and (14.4)}], \\ \frac{d\pi}{dz} &= \frac{\partial \pi}{\partial w} \frac{dw}{dz} + \frac{\partial \pi}{\partial N} \frac{dN}{dz} + \frac{\partial \pi}{\partial z} = \frac{\partial \pi}{\partial z} = -f' e' N < 0. \end{aligned}$$

14.4 Labor Discipline with Profit Signaling

- (a) trivial;
- (b) Treat w_H as a function of w_L , and differentiate the participation constraint, getting

$$p_h u'(w_H) \frac{dw_H}{dw_L} + (1 - p_h) u'(w_L) = 0.$$

Thus

$$\frac{dw_H}{dw_L} = -\frac{1 - p_h}{p_h} \frac{u'(w_L)}{u'(w_H)} < -\frac{1 - p_h}{p_h} < 0. \quad (14.5)$$

The second inequality (which we use later) holds because $w_L < w_H$, so if the agent is strictly risk averse, u' is decreasing. The participation constraint is thus decreasing. Now take the derivative of equation (14.5), getting

$$\frac{d^2 w_H}{dw_L^2} = -\frac{1 - p_h}{p_h} \left[\frac{u''(w_H)}{u'(w_H)} - \frac{u'(w_L) u''(w_H)}{u'(w_H)^2} \frac{dw_H}{dw_L} \right] > 0.$$

Thus the participation constraint is convex.

- (c) We differentiate the incentive compatibility constraint $u(w_H) = u(w_L) + \text{constant}$, getting

$$u'(w_H) \frac{dw_H}{dw_L} = u'(w_L),$$

so $dw_H/dw_L > 1 > 0$ and the incentive compatibility constraint is increasing. Differentiate again, getting

$$u''(w_H)\frac{dw_H}{dw_L} + u'(w_H)\frac{d^2w_H}{dw_L^2} = u''(w_L).$$

Thus

$$u'(w_H)\frac{d^2w_H}{dw_L^2} = u''(w_L) - u''(w_H)\frac{dw_H}{dw_L} < u''(w_L) - u''(w_H) < 0,$$

and the constraint is concave.

- (d) The slope of the iso-cost line is $|dw_H/dw_L| = (1 - p_h)/p_h$, which is less than the slope of participation constraint, which is

$$|(1 - p_h)u'(w_L)/p_h u'(w_H)|,$$

by equation by equation (14.5).

- (e) Obvious from the above;

14.5 Peasants and Landlords

- (a) First, w_H and w_L must satisfy a *participation constraint*: w_H and w_L must be sufficiently large that the peasant is willing to work at all. Suppose the peasant's next best alternative gives utility z . Then the landowner must choose w_H and w_L so that the peasant's expected utility is at least z :

$$p_h u(w_H) + (1 - p_h)u(w_L) - d_h \geq z. \quad (\text{PC})$$

Second, w_H and w_L must satisfy an *incentive compatibility constraint*: the payoff (i.e., the expected return) to the peasant for working hard must be at least as great as the payoff to not working hard. Thus we must have

$$p_h u(w_H) + (1 - p_h)u(w_L) - d_h \geq p_l u(w_H) + (1 - p_l)u(w_L) - d_l.$$

We can rewrite this second condition as

$$[u(w_H) - u(w_L)](p_h - p_l) \geq d_h - d_l. \quad (\text{ICC})$$

- (b) The problem is to minimize $p_h w_H + (1 - p_h)w_L$ subject to PC and ICC. This is the same as maximizing $-p_h w_H - (1 - p_h)w_L$ subject to the same constraints, so we form the Lagrangian

$$\begin{aligned}\mathcal{L}(w_H, w_L, \lambda, \mu) = & -p_h w_H - (1 - p_h)w_L \\ & + \lambda[p_h u(w_H) + (1 - p_h)u(w_L) - d_h - z] \\ & + \mu[(u(w_H) - u(w_L))(p_h - p_l) - (d_h - d_l)].\end{aligned}$$

The first order conditions can be written:

$$\mathcal{L}_H = 0, \mathcal{L}_L = 0, \quad \lambda, \mu \geq 0;$$

if $\lambda > 0$ then the PC holds with equality;

if $\mu > 0$, then the ICC holds with equality.

But we have

$$\begin{aligned}\mathcal{L}_H &= -p_h + \lambda p_h u'(w_H) + \mu u'(w_H)(p_h - p_l) = 0, \\ \mathcal{L}_L &= -1 + p_h + \lambda(1 - p_h)u'(w_L) - \mu u'(w_L)(p_h - p_l) = 0.\end{aligned}$$

We show that $\lambda = 0$ is impossible. Assume the contrary, so $\lambda = 0$. Then by adding the two first order condition's, we get

$$\mu(u'(w_H) - u'(w_L))(p_h - p_l) = 1,$$

which implies $u'(w_H) > u'(w_L)$, so $w_H < w_L$ (by declining marginal utility of income). This is of course silly, since the peasant will not want to work hard if the wage given high profits is less than the wage give low profits! More formally, ICC implies $u(w_H) > u(w_L)$, so $w_H > w_L$. It follows that our assumption that $\lambda = 0$ was contradictory, and hence $\lambda > 0$. So PC holds as an equality.

We now show that $\mu = 0$ is impossible. Assume the contrary, so $\mu = 0$. Then the first order condition's $\mathcal{L}_H = 0$ and $\mathcal{L}_L = 0$ imply $u'(w_H) = 1/\lambda$ and $u'(w_L) = 1/\lambda$. Since $u'(w_H) = u'(w_L) = 1/\lambda$, $w_H = w_L$ (because u' is strictly decreasing). This also is impossible by the ICC. Hence $\mu > 0$ and the ICC holds as an equality.

- (c) Now suppose the landlord has concave utility function v , with $v' > 0$ and $v'' < 0$. The peasant is risk neutral, so we can assume her utility function is $u(w, d) = w - d$ where w is income and d is effort. The assumption that high effort produces a surplus means that the following social optimality (SO) condition holds:

$$p_h H + (1 - p_h)L - d_h > p_l H + (1 - p_l)L - d_l,$$

or

$$(p_h - p_l)(H - L) > d_h - d_l. \quad (\text{SO})$$

The landlord wants to maximize

$$p_h v(H - w_H) + (1 - p_h)v(L - w_L)$$

subject to the participation constraint

$$p_h w_H + (1 - p_h)w_L - d_h \geq z \quad (\text{PC})$$

and the incentive compatibility constraint, which as before reduces to

$$(p_h - p_l)(w_h - w_L) \geq d_h - d_l. \quad (\text{ICC})$$

Form the Lagrangian

$$\begin{aligned} \mathcal{L} = & p_h v(H - w_H) + (1 - p_h)v(L - w_L) \\ & + \lambda(p_h w_H + (1 - p_h)w_L - d_h - z) \\ & + \mu((p_h - p_l)(w_h - w_L) - (d_h - d_l)), \end{aligned}$$

so the first order condition's are $\partial \mathcal{L} / \partial w_H = \partial \mathcal{L} / \partial w_L = 0$, $\lambda, \mu \geq 0$, $\lambda > 0$ or the PC holds as an equality, and $\mu > 0$ or the ICC holds as an equality. The conditions $\partial \mathcal{L} / \partial w_H = \partial \mathcal{L} / \partial w_L = 0$ can be written as

$$\frac{\partial \mathcal{L}}{\partial w_H} = -p_h v'(H - w_H) + \lambda p_h + \mu(p_h - p_l) = 0 \quad (\text{FOC1})$$

$$\mu(p_h - p_l) = 0 \quad (\text{FOC2}).$$

We first show that the PC holds as an equality by showing that $\lambda = 0$ is impossible. Suppose $\lambda = 0$. Then (FOC) gives

$$\begin{aligned} -p_h v'(H - w_H) + \mu(p_h - p_l) &= 0 \\ -(1 - p_h)v'(L - w_L) - \mu(p_h - p_l) &= 0. \end{aligned}$$

If $\mu > 0$, this says that $v'(L - w_L) < 0$, which is impossible. If $\mu = 0$, this says that $v'(L - w_L) = 0$, which is also impossible. Thus $\lambda = 0$ is impossible, and the PC holds as an equality.

Now we can rewrite FOC1 and FOC2 as

$$\begin{aligned} v'(H - w_H) - \lambda &= \mu(p_h - p_l)/p_h \\ v'(L - w_L) - \lambda &= -\mu(p_h - p_l)/(1 - p_h). \end{aligned}$$

If $\mu > 0$, then $v'(H - w_H) > v'(L - w_L)$, so $H - w_H < L - w_L$, or $H - L < w_H - w_L$. But $\mu > 0$ implies that the ICC holds as an equality, so $(p_h - p_l)(w_H - w_L) = d_h - d_l$, and $d_h - d_l < (p_h - p_l)(H - L)$ from SO, implying $H - L > w_H - w_L$. This is a contradiction, and hence $\mu = 0$; i.e., there is no optimum in which the ICC holds as an equality!

What is an optimum? Well, if $\mu = 0$, equations 14.6 and 14.6 imply that $H - w_h = L - w_L$, since v' is strictly increasing (because the landlord is risk averse). This means the landlord gets a fixed rent, which proves the theorem.

14.6 Mr. Smith's Car Insurance

- (a) If he is careful, the value is $(1/2)\ln(1201) - \epsilon$, and if he is careless the value is $(1/4)\ln(1201)$. Being careful is worthwhile as long as $(1/2)\ln(1201) - \epsilon \geq (1/4)\ln(1201)$, or

$$\epsilon \leq \ln(1201)/4 = 1.77.$$

- (b) If he buys the insurance, his utility is $\ln(1201 - x)$ if he is careless, and $\ln(1201 - x) - \epsilon$ if he is careful. Thus he will not be careful. The probability of theft is then 75%, and since the insurance company must give a fair lottery to Mr. Smith, $x = (0.75)1200 = 900$.
- (c) The expected value of the car plus insurance is $\ln(1201 - 900) = \ln(301) = 5.707$.

- (d) This is preferable to being careless without insurance, which has value

$$(1/4) \ln(1201) = 1.77.$$

The value of being careful without insurance has value $(1/2) \ln(1201) - \epsilon < (1/2) \ln(1201) = 3.55$, so Mr. Smith should buy the insurance policy whether or not he would be careful without it.

- (e) Since Mr. Smith is careful without insurance, we know $\epsilon \leq 1.77$. Since the insurance company's lottery is fair, we have $x = (1200 - z)/2 = 600 - z/2$ if Mr. Smith is careful, and $x = 3(1200 - z)/4 = 900 - 3z/4$ if Mr. Smith is careless. If Mr. Smith is careless, the value of car plus insurance is

$$\begin{aligned} \frac{1}{4} \ln(1201 - x) + \frac{3}{4} \ln(1201 - z - x) = \\ \frac{1}{4} \ln(301 + \frac{3z}{4}) + \frac{3}{4} \ln(301 - \frac{z}{4}). \end{aligned}$$

The derivative of this with respect to z is

$$(3/16)/(301 + 3z/4) - (3/16)/(301 - z/4) < 0,$$

so the optimal deductible is zero. Then $x = 900$, and the value of car plus insurance is $\ln(1201 - 900) = \ln(301)$. If Mr. Smith is careful, then the value of car plus insurance is

$$\begin{aligned} \frac{1}{2} \ln(1201 - x) + \frac{1}{2} \ln(1201 - z - x) = \\ \frac{1}{2} \ln(601 + \frac{z}{2}) + \frac{1}{2} \ln(601 - \frac{z}{2}). \end{aligned}$$

derivative of this with respect to z is $(1/8)/(601 + z/2) - (1/8)/(601 - z/2) < 0$, so again the optimal deductible is zero. But z must be sufficiently large that Mr. Smith wants to be careful, or the insurance company will not be willing to issue the insurance at the low rate $x = 600 - z/2$. To be worthwhile to be careful, we must have

$$\begin{aligned} \frac{1}{2} \ln(601 - \frac{z}{2}) + \frac{1}{2} \ln(601 - \frac{z}{2}) - \epsilon \geq \\ \frac{1}{4} \ln(601 + \frac{z}{2}) + \frac{3}{4} \ln(601 - \frac{z}{2}), \end{aligned}$$

$\ln(601 + z/2) - \ln(601 - z/2) \geq 4\epsilon$. The minimum z satisfying this is when the equality holds, and we have

$$z = \frac{4(e^{4\epsilon} - 1)}{e^{4\epsilon} + 1} > 0.$$

- (f) With insurance, the expected value of the lottery for Mr. Smith is unchanged, but the risk has decreased. Since he is risk averse, he is better off with the insurance than without.

14.7 A Generic One-Shot Principal Agent Model

- (a) Intuitively, we argue as follows. Suppose the principal wants to induce the agent to choose action a_k . If the participation constraint is not binding, we can reduce all the payments $\{w_{kj}\}$ by a small amount without violating either the participation or the incentive compatibility constraints. Moreover, if none of the incentive compatibility constraints is binding, then the payoff system $\{w_{kj}\}$ is excessively risky, in the sense that the various $\{w_{kj}\}$ can be ‘compressed’ around their expected value without violating the incentive compatibility constraint.

Formally, we form the Lagrangian,

$$\begin{aligned} \mathcal{L} = & \pi(a_k) - \mathbf{E}_k w_k + \lambda[\mathbf{E}_k u(w_k) - d(a_k) - u_0] \\ & + \sum_{\substack{i=1 \\ i \neq k}}^n \mu_i \{[\mathbf{E}_k u(w_k) - d(a_k)] - [\mathbf{E}_i u(w_k) - d(a_i)]\}, \end{aligned}$$

where $\mathbf{E}_i$ means take the expectation with respect to probabilities $\{p_{i1}, \dots, p_{im}\}$, λ is the Lagrangian multiplier for the participation constraint, and μ_i is the Lagrangian multiplier for the i^{th} incentive compatibility constraint. Writing the expectations out in full (in ‘real life’ you wouldn’t do this, but it’s valuable for pedagogical purposes), we get

$$\begin{aligned} \mathcal{L} = & \pi(a_k) - \sum_{j=1}^m p_{kj} w_{kj} + \lambda \left[\sum_{j=1}^m p_{kj} u(w_{kj}, a_k) - u_0 \right] \\ & + \sum_{i=1}^n \mu_i \left[\left(\sum_{j=1}^m p_{kj} u(w_{kj}, a_k) \right) - \left(\sum_{j=1}^m p_{ij} u(w_{kj}, a_i) \right) \right]. \end{aligned}$$

The Kuhn-Tucker conditions for the problem assert that at a maximum,

- i. $\frac{\partial \mathcal{L}}{\partial w_{kj}} = 0$ for $j = 1, \dots, m$,
- ii. $\lambda, \mu_1, \dots, \mu_n \geq 0$,
- iii. if $\lambda > 0$, then the participation constraint is binding (i.e., holds as an equality);
- iv. if $\mu_i > 0$ for some $i = 1, \dots, n$, then the incentive compatibility constraint holds for action a_i .

In our case, we have

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial w_{kj}} &= -p_{kj} + \lambda p_{kj} u'(w_{kj}, a_k) \\ &+ \sum_{i=1}^n \mu_i (p_{kj} - p_{ij}) u'(w_{kj}, a_k) = 0 \quad j = 1, \dots, m. \end{aligned}$$

Collecting terms, we have

$$\left(\lambda p_{kj} + \sum_{i=1}^n \mu_i (p_{kj} - p_{ij}) \right) = \frac{p_{kj}}{u'(w_{kj}, a_k)} \quad j = 1, \dots, m. \quad (14.6)$$

Now we sum this equation from $j = 1$ to m , noting that $\sum_j p_{ij} = 1$ for all i , getting

$$\lambda = \sum_{j=1}^m \frac{p_{kj}}{u'(w_{kj}, a_k)} > 0,$$

which proves that the participation constraint is binding.

- (i) To see that at least one incentive compatibility constraint is binding suppose all the μ_i are zero. Then equation (14.6) gives $\lambda = 1/u'(w_{kj}, a_k)$ for all j . If the agent is risk averse, this implies all the w_{kj} are equal (since risk aversion implies strictly concave preference, which implies strictly monotonic marginal utility), which means a_k must be the agent's most preferred action.
- (ii) If the parties could write an enforceable contract, then only the participation constraint would be relevant. To induce the agent to perform a_i , the principal would pay w_i^* where $u(w_i^*, a_i) = u_0$. The principal will then choose action a_l such that $\pi(a_l) - w_l^*$ is a maximum.

Suppose the optimal action when no contract can be written is a_k , and the wage structure is $w_k(s)$. The participation

constraint is binding, so $\mathbf{E}_k u(w_k) = d(a_k) + u_0$. Since a_k is not the agent's unconstrained preferred action (by assumption), $w_k(s)$ is not constant, and since the agent is strictly risk averse, we have

$$u(w_k^*, a_k) = u_0 = \mathbf{E}u(w_k, a_k) < u(\mathbf{E}_k w_k, a_k),$$

so $w_k^* < \mathbf{E}_k w_k$. Since $\pi(a_l) - w_l^* \geq \pi(a_k) - w_k^* > \pi(a_k) - \mathbf{E}_k w_k$, we are done.

- (b) Suppose again that a_l is the principal's optimal choice when an enforceable contract can be written. Then a_l maximizes $\pi(a_i) - w_i^*$ where $u(w_i^*, a_i) = u_0$. If the agent is risk neutral, suppose the agent receive $\pi(a)$ and pays the principal the fixed amount $\pi(a_l) - w_l^*$. We can assume $u(w, a) = w + d(a)$ (why?). The agent then maximizes

$$\begin{aligned} \pi(a_i) - d(a_i) &= u(\pi(a_i)) - u(w_i^*) - u_0 \\ &= \pi(a_i) - w_i^* - u_0, \end{aligned}$$

which of course occurs when $i = l$. This proves the theorem. Q.E.D.